e-Pathshala पाठशाला
A Gateway to All Post Graduate Courses

An MHRD Project under its National Mission on Education through ICT (NME-ICT)

Subject: Physics

Production of Courseware

e-Content for Post Graduate Courses

Paper No. : Classical Mechanics

Module : Constraints and Generalised Coordinates

Development Team

Principal Investigator	Prof. Vinay Gupta, Department of Physics and Astrophysics, University of Delhi, Delhi
Paper Coordinator	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Writer	Ashok Goyal, Department of Physics and Astrophysics, University of Delhi, Delhi
Content Reviewer	

Description of Module	
Subject Name	Physics
Paper Name	Classical Mechanics
Module Name/Title	Constraints and Generalised Coordinates
Module Id	1

Constraints and Generalized Coordinates

Contents:

1. Learning Objectives :

- ❖ Learn to classify constraints
- ❖ Learn to use constraints to identify degrees of freedom needed to specify a dynamical system.
- ❖ Use of generalized coordinates to simplify the description of a dynamical system.

1. Introduction:

Classical mechanics deals with the description of a dynamical system which evolves in a three dimensional Euclidean space. The space and time are considered absolute and immutable entities. In the Newtonian world view, motion is described by Newton's laws of motion which are valid in an inertial frame of reference. The dynamical system may be a point particle, a rigid body or a collection of particles. Classical mechanics was developed by Newton. An alternative and attractive formulation was developed by Lagrange, Euler, Hamilton, Poisoon, Jacobi and others. This formulation made the transition from classical mechanics to quantum mechanics easier and found important role in the theory of classical and quantum fields.

2. Constraints

In real physical situations the motion is often constrained to move in a way such that some of its coordinates or velocity components satisfy certain relations all through its motion. The relations can be expressed in the form of equations or inequalities. For example, the motion of a particle on a circle or on an ellipse in the X-Y plane satisfies

$$x^2 + y^2 = a^2 \qquad (1.1)$$

or, $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1.2)$$

if it is moving on a circle of radius a or on an ellipse of semi-major axis a and semi-minor axis b. The coordinates of a particle confine to move within a sphere of radius a satisfies

$$x^2 + y^2 + z^2 \leq a^2 \qquad (1.3)$$

The constrained motion results on account of certain forces called 'constraining forces' which arise when the particle is in contact with the surface or the curve on which it is constrained to move. Constraints can be classified in different classes depending on their nature **'Holonomic'** constraints can be expressed in terms if algebraic equations in the form

$$f(\boldsymbol{r_1}, \boldsymbol{r_2} \dots \dots \dots \dots \dots, t) = 0 \qquad (1.4)$$

and they can be made independent of velocities. If the constraints depend on velocities and cannot be expressed in **'integrable equations'** they are termed **'Non holonomic'.** Non-holonomic constraints may involve frictional forces or expressed in terms of an inequality or in terms of non-integrable equations like $xdy - ydx = cx^2$. The constraints that do not depend on time explicitly are called **'Scleronomic'** and **Rheonomic**' if they depend explicitly on time. Further the constraints are said to be 'conservative' if the total mechanical energy during the motion is conserved and 'dissipative' when the constraint forces do work and the mechanical energy is not conserved.

Examples:

1. Constraint equation satisfied by a particle moving on the surface of a sphere of radius a is
$$x^2 + y^2 + z^2 = a^2$$

2. Constraints satisfied by the motion of an atom of radius a moving in a rectangular cavity of size are
$$a \le x \le l - a; \;\; b \le y \le b - a; \;\; a \le z \le \omega - a$$

In view of the fact that the coordinates of a constrained system satisfy certain relations, the number of independent variables required to fix the position and configuration of a dynamical system is reduced in comparison to the unconstrained or free motion. The number of independent variables $(u_1, u_2, \dots \dots \dots \dots . u_n)$ required for the descripton of the constrained system is called the number of **'degrees of freedom'** of the system

1. A free particle moving in a three dimensional Euclidean space has three degrees of freedom (x, y, z) in Cartesian coordinates each varying between $-\infty$ to $+\infty$. N free particles likewise have 3N degrees of freedom.
2. A diatomic molecule having two identical atoms joined together by electromagnetic forces such that the bond length remains fixed has 3 x 2 – 1=5 degrees of freedom. Each atom has 3 degrees of freedom with one constraint $|\boldsymbol{r_1} - \boldsymbol{r_2}| = l$
3. A rigid body : A Rigid body is mathematically idealized as a system with large number of particles with fixed distances among themselves has 6 degrees of freedom. This is because any three points in a rigid body that are not collinear have 3 x 2 = 6 degrees of freedom and once any three arbitrary points in the rigid body are fixed, the rigid body is completely fixed.

How to choose the degrees of freedom?

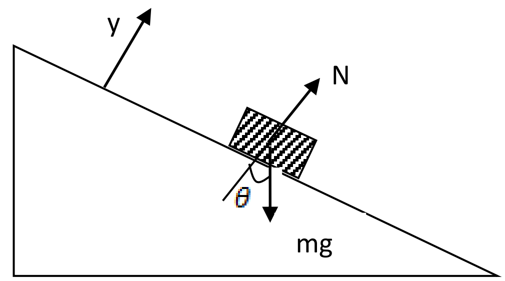
There is a choice in choosing degrees of freedom in terms of

i) choice of the origin

ii) coordinate system (can be Cartesian, cylindrical, spherical etc.)

Simple Example of constrained motion

Example 1. A Block sliding without friction on an inclined plane



Coordinate system: X-axis is pointing along the surface of the plane, Y-axis normal to the plane. The Block is confined to slide without friction along the plane, therefore the constraint is y = constant ne motion along Y-axis. The equations of motion are

$$m\ddot{x} = mg\sin\theta$$
$$m\ddot{y} = N - mg\cos\theta = 0$$

Thus the constraining force acts normal to the plane and is given by $N = mg\cos\theta$

Integrating (1.5)

$$\ddot{x} = g\sin\theta$$
$$\dot{x} = \dot{x}(0) + g\sin\theta\ t$$
$$x(t) = x(0) + \int (\dot{x}(0) + g\sin\theta\ t)\,dt$$
$$= x(0) + \dot{x}(0)t + \frac{1}{2}gt^2\sin\theta$$

Let $x(0) = \dot{x}(0) = 0$ i.e the blocks starts at rest at the top.

$$\therefore x(t) = \frac{1}{2}gt^2\sin\theta$$

Example 2. Atwoods's Machine.

Two masses connected by a mass less inextensible string of fixed length passing over a frictionless pulley.

Coordinate system: Since there is only vertical motion we use Y – coordinate in the down direction.

Equation of motion of m_1 and m_2 are given by

$$m_1\ddot{y}_1 = -m_1 g + T;\ m_2\ddot{y}_2 = -m_2 g + T \tag{1.7}$$

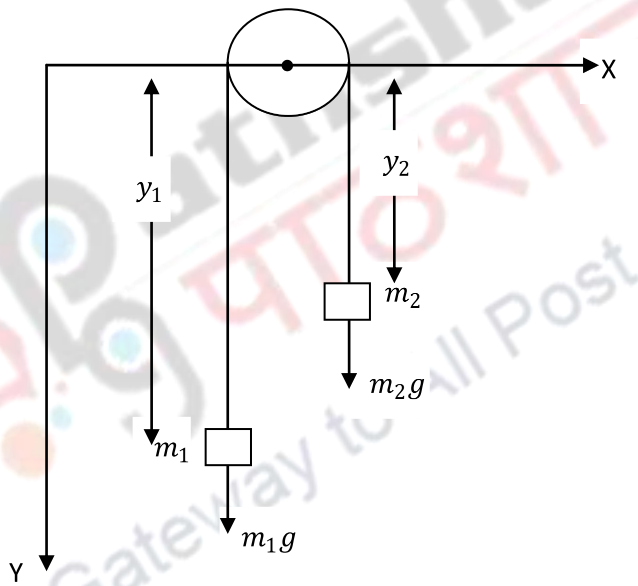
Where T is the tension in the string. Since the length of the string is fixed

$$y_1 + y_2 = -l \qquad i.e.\quad y_2 = -l - y_1 \tag{1.8}$$

$$\text{So}\ \ \dot{y}_1 = -\dot{y}_2 \qquad and \qquad \ddot{y}_1 = -\ddot{y}_2 \tag{1.9}$$

$$\therefore m_1\ddot{y}_1 = -m_1 g + m_2(\ddot{y}_2 + g)$$

$$(m_1 + m_2)\ddot{y}_1 = (-m_1 + m_2)g$$



$$\ddot{y}_1 = -\frac{(m_1 - m_2)}{m_1 + m_2} g \qquad and \quad T = \frac{2m_1 m_2}{m_1 + m_2} \tag{1.10}$$

Generalised cordinates:

We saw that for a system of N particles if we have k independent constraint equations, the number of degrees of freedom is 3N – k. The dynamical behavior of the system depends only on the coordinates corresponding to the degrees of freedom. Since these are fewer degrees of freedom then the position coordinates, required to specify the position of each particle in the system, can we eliminate the unnecessary coordinates? This will undoubtedly simplify the analysis of the motion particularly if we choose coordinates that take into account the constraints and are independent.

For holonomic constraints it is indeed possible to define a set of 3N-k independent coordinates called 'Generalized Coordinates' $\{q_k\}$ that specify the motion of the system subject to the given constraints and which are independent of each other. The Cartesian coordinates can then be expressed in terms of known function of generalized coordinates q_k i.e.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . q_n, t) \tag{1.11}$$

In terms of components

$$\begin{aligned} x_1 &= x_1(q_1, q_2, \ldots \ldots \ldots \ldots \ldots . q_n, t) \\ x_2 &= x_2(q_1, q_2, \ldots \ldots \ldots \ldots \ldots . q_n, t) \\ &\vdots \\ &\vdots \\ x_{3N} &= x_{3N}(q_1, q_2, \ldots \ldots \ldots \ldots \ldots . q_n, t) \end{aligned} \tag{1.12}$$

where $n = 3N - k$. It may or may not be always possible to have an analytical expression for these functions. The generalized coordinates eliminate the constraints forces from the problem and they are independent. The generalized coordinates need not be Cartesian and can be chosen as suitable to the problem considered. For example, the motion of a particle moving on an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.13}$$

has only one degree of freedom which can be chosen to be the polar angle θ and the particle coordinates X and Y are expressible in terms of the generalized coordinates θ as

$$x = a\cos\theta\,, \quad y = b\sin\theta \tag{1.14}$$

for which the constraint equation is automatically satisfied.

Just as the velocity components $v_i = \frac{d}{dt} x_i$ are defined for poition coordinates, we can define generalized velocity

$$\dot{q_k} = \frac{dk_k}{dt} \qquad (1.15)$$

Velocities being defined as total time derivative of the said coordinate. For a function $f(\{q_k\},t)$ the total time derivative is given by the chain rule:

$$\frac{d}{dt} f(\{q_k\},t) = \sum_k \frac{\partial f}{\partial q_k} \frac{dq_k}{dt} + \frac{\partial f}{\partial t} \qquad (1.16)$$

The use of generalized coordinates in place of position coordinates of the system was to eliminate the nondynamical degrees of freedom from the system. In a similar manner, constraint forces can be eliminated leaving only the generalized forces.

Let $F_i(\boldsymbol{r}_i)$ be the force acting on the ith particle of a N particle system. The work done by the force for an arbitrary displacement $\delta \boldsymbol{r}_i$ is

$$\boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i$$

And the total work done on the system by all the forces is

$$\delta W = \sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i \qquad (1.17)$$

Now $\boldsymbol{r}_i$'s are expressible in terms of generalized coordinates as

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \dots \dots \dots q_n) \qquad (1.18)$$

Then

$$\delta \boldsymbol{r}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j \qquad (1.19)$$

Substituting in (1.17)

$$\delta W = \sum_{i=1}^{N} \boldsymbol{F}_i(\boldsymbol{r}_i) \cdot \sum_{j=1}^{r} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \delta q_j = \sum_j \delta q_j \left(\sum_i \boldsymbol{F}_i(\boldsymbol{r}_i) \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \right) \qquad (1.20)$$

Define

$$\Im_j = \sum_i \boldsymbol{F}_i(\boldsymbol{r}_i) \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j} \qquad (1.21)$$

as the Generalized force and the total work done on the system can now be written as

$$\delta W = \sum_j \Im_j \cdot \delta q_j \tag{1.22}$$

expressed only in terms of the generalized coordinates and forces. The expression $\sum_j \Im_j \cdot \delta q_j$ denotes the work done by all generalized forces which equal to the number of degrees of freedom of the system in the arbitrary displacement in generalized coordinates. Let us illustrate this with the example of Atwood's machine.

Choose the coordinates as shown y_1 and y_2 are the positions of masses m_1 and m_2 from a horizontal plane passing through the pulley. The y-axis points upwards since the length of the string is fixed and does not charge with time.

$$l = -y_1 - y_2 \tag{1.23}$$

We can choose either y_1 or y_2 as generalized

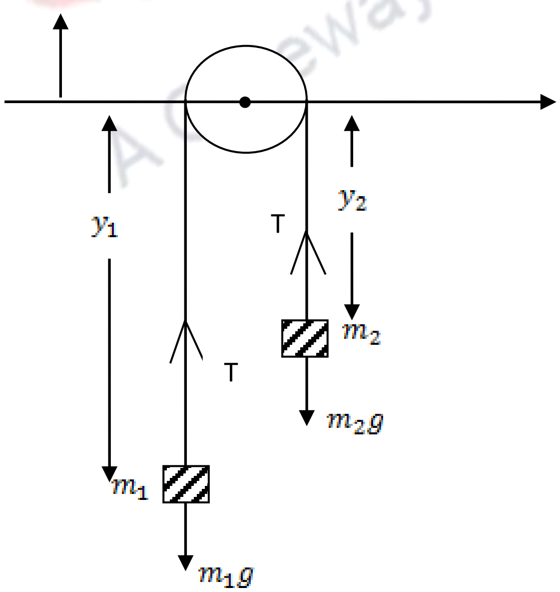
Let us choose $y_1 = y$ as the generalized coordinates generalized velocity is given by

The Force $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ acting on masses $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ are

$$\boldsymbol{F_1} = (-m_1 g y_1 + T)\hat{y} = (-m_1 g y + T)\hat{y}$$

$$\boldsymbol{F_2} = (-m_2 g y_2 + T)\hat{y} = (-m_2 g(-l - y) + T)\hat{y}$$

Now



$$\delta W = \boldsymbol{F_1} \cdot \delta y_1 \hat{y} + \boldsymbol{F_2} \cdot \delta y_2 \hat{y} \qquad \delta y_1 = \delta y$$

$$= (\boldsymbol{F_1} - \boldsymbol{F_2})\delta y \hat{y} \qquad \delta y_2 = -\delta y$$

$$= -g(m_1 - m_2)\delta y$$

$\therefore$ generalized force $\Im_y = -g(m_1 - m_2)$.

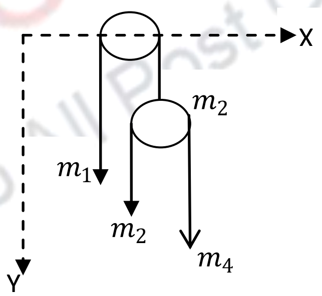
Problem 1: How many degrees of freedom does the double pendulum has? Write down the constraint equations.

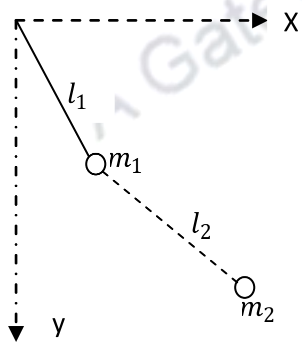
Ans. 2, $x_1^2 + y_1^2 = l_1^2$, $(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$

Hint: Let (x_1, y_1) and (x_2, y_2) be the coordinates of m_1 and m_2.

Problem 2. Four masses m_1, m_2, m_3 and m_4 are hanging such that they can move only in the vertical direction. How many degrees of freedom does it have? Write down the constraint equations.

[Hint: Let y_1, y_2, y_3, y_4 be the vertical distances. The constraints are $y_1 + y_2 = l_1$; $y_2 + y_4 = l_2$ hence two independent coordinates and two degrees of freedom.





Double Pendulum (Prob.1) (Prob.2)

Summary :

- The constraints which can be expressed as algebraic or integrable differential equations are holonomic.
- If a halonomic constraint depends on time explicitly it is called Rhenomic, if there is no explicit time dependence, they are called Scleronomic.
- Non-holonomic constraints involve frictional forces or are expressed in the form of inequalities.
- Number of independent variables required for the description of a system is called the number of degrees of freedom.